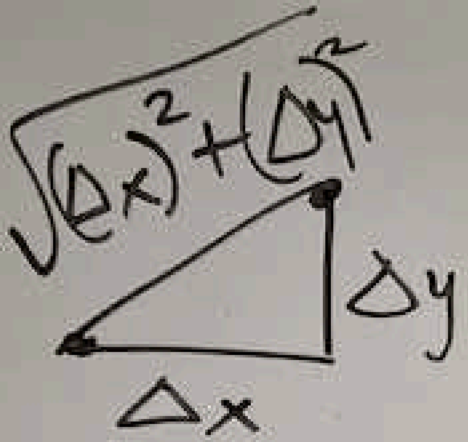


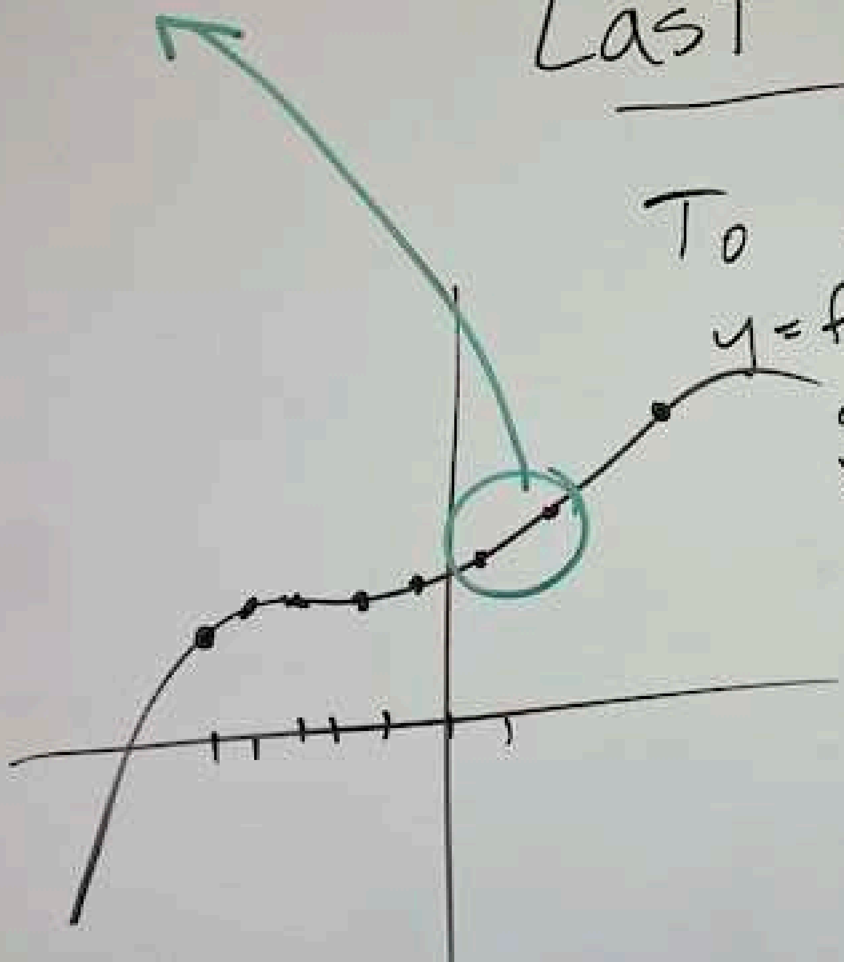
Last time:

To find the length of a curve:

$y = f(x)$

or

$x = g(y)$



$$\text{Length} \approx \sum \sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$= \int_a^b \sqrt{dx^2+dy^2}$$

$$\left(\sqrt{\frac{dx^2+dy^2}{dx^2}}\,dx\right)$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

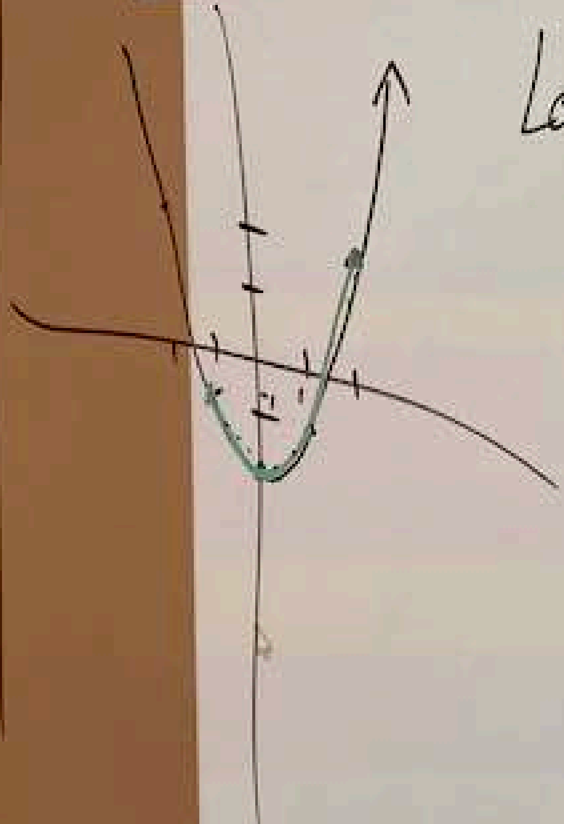
$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

or

$$\sqrt{\frac{dx^2+dy^2}{dy^2}}\;dy \quad = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2+1}\;dy$$

Most of the time, yucky, so compute numerically.

$\boxed{\text{Ex}}$ $y = x^2 - 2$. Find the length of the curve from $(-1,-1)$ to $(2,2)$.



$$\text{Length} = \int_{x=-1}^{x=2} \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx$$

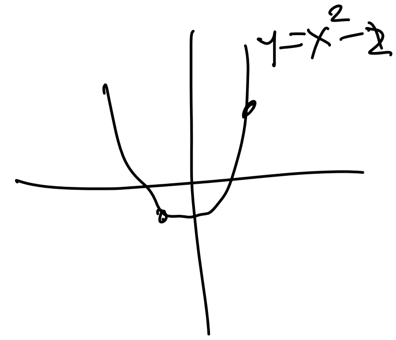
$$\frac{dy}{dx} = 2x$$

$$= \int_{-1}^{2} \sqrt{4x^2 + 1}\, dx$$

Let $2x = \tan\theta$

$$\Rightarrow 2\,dx = \sec^2\theta\, d\theta$$

To compute the length



$$\int_{-1}^{2} \sqrt{4x^2+1}\,dx$$

$2x = \tan\theta$

$2\,dx = \sec^2\theta\,d\theta$

$$\int_{x=-1}^{2} \underbrace{\sqrt{\tan^2\theta+1}}_{\sec\theta}\ \frac{1}{2}\sec^2\theta\,d\theta$$

$$= \int_{x=-1}^{2} \frac{1}{2}\sec^3\theta\,d\theta = \frac{1}{2}\int_{x=-1}^{2} \frac{1}{\cos^3\theta}\,d\theta$$

$$= \frac{1}{2}\int_{x=-1}^{2} \frac{\cos\theta\,d\theta}{(\cos^2\theta)^2} = \frac{1}{2}\int_{x=-1}^{2} \frac{\cos\theta\,d\theta}{(1-\sin^2\theta)^2}$$

let $u = \sin\theta$ $\quad du = \cos\theta\,d\theta$

$$= \frac{1}{2}\int_{x=-1}^{2} \frac{du}{(1-u^2)^2}$$

$$\frac{1}{(1+u)^2(1-u)^2} = \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C}{(1-u)} + \frac{D}{(1-u)^2}$$

$$1 = A(1+u)(1-u)^2 + B(1-u)^2 + C(1-u)(1+u)^2 + D(1+u)^2$$

$u = 1 \Rightarrow 1 = D \cdot 4 \Rightarrow D = \frac{1}{4}$

$u = -1 \Rightarrow 1 = B \cdot 4 \Rightarrow B = \frac{1}{4}$

$\underline{u^3 \text{ term:}}$ $0 = A - C \Rightarrow A = C$

$\underline{\text{constant term}}$ $1 = A + B + C + D$

$1 = A + \frac{1}{4} + A + \frac{1}{4}$

$\frac{1}{2} = 2A$

$\Rightarrow A = \frac{1}{4}$

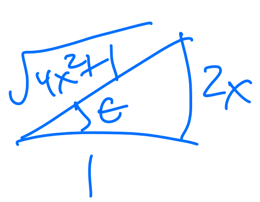
$C = \frac{1}{4}$.

$$= \frac{1}{8} \int_{x=-1}^{2} \left[\frac{1}{1+u} + \frac{1}{(1+u)^2} + \frac{1}{(1-u)} + \frac{1}{(1-u)^2} \right] du$$

$(1+u)^{-2}$

$$= \frac{1}{8} \ln|1+u| - \frac{1}{1+u} - \ln|1-u| + \frac{1}{1-u} \Big|_{x=-1}^{2}$$

$u = \sin\theta$



$2x = \tan\theta$

$$\sin\theta = \frac{2x}{\sqrt{4x^2+1}} = u$$

$$= \frac{1}{8}\ln\left|1+\frac{2x}{\sqrt{4x^2+1}}\right| - \frac{1}{1+\frac{2x}{\sqrt{4x^2+1}}} - \ln\left|1-\frac{2x}{\sqrt{4x^2+1}}\right| + \frac{1}{1-\frac{2x}{\sqrt{4x^2+1}}}\Bigg|_{-1}^{2}$$

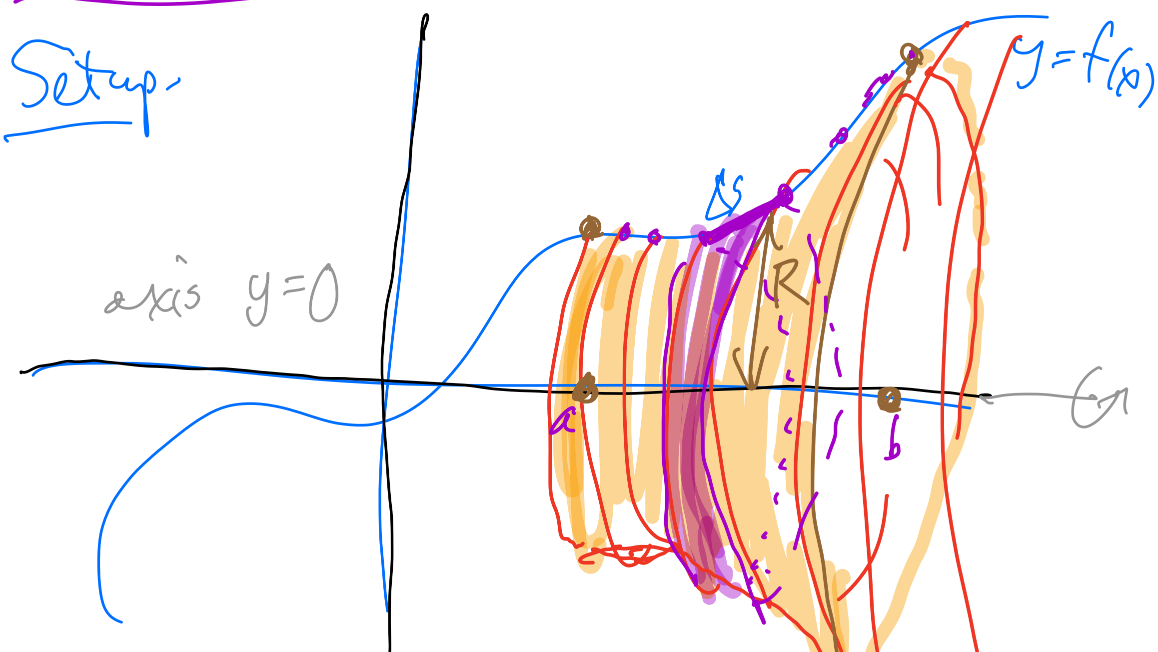
$$= \boxed{\frac{1}{8}\ln\left(1+\frac{4}{\sqrt{17}}\right) - \frac{1}{1+\frac{4}{\sqrt{17}}} - \ln\left(1-\frac{4}{\sqrt{17}}\right) + \frac{1}{1-\frac{4}{\sqrt{17}}} - \left[\frac{1}{8}\ln\left(1+\frac{-2}{\sqrt{5}}\right) - \frac{1}{1-\frac{2}{\sqrt{5}}} - \ln\left(1+\frac{2}{\sqrt{5}}\right) + \frac{1}{1+\frac{2}{\sqrt{5}}}\right]}$$

exact length of our curve between $x=-1$ & $x=2$.

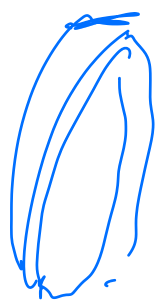
Next application of the integral

Calculating surface area of a surface of revolution.

Setup.



Surface Area of slice $=$



$$= 2\pi R \sqrt{\Delta x^2 + \Delta y^2}$$

in terms of x or y

integral = total surface area $= \displaystyle\int_a^b 2\pi R \overbrace{\sqrt{dx^2 + dy^2}}^{ds}$

Usually a very hard integral to calculate

→ need numerical calc.